

Iteration Index of a Zero Forcing Set in a Graph

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Abstract

Let each vertex of a graph $G = (V(G), E(G))$ be given one of two colors, say, “black” and “white”. Let Z denote the (initial) set of black vertices of G . The *color-change rule* converts the color of a vertex from white to black if the white vertex is the only white neighbor of a black vertex. The set Z is said to be a *zero forcing set* of G if all vertices of G will be turned black after finitely many applications of the color-change rule. The *zero forcing number* of G is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$. Zero forcing parameters have been studied and applied to the minimum rank problem for graphs in numerous articles. We define the *iteration index of a zero forcing set of a graph G* to be the number of (global) applications of the color-change rule required to turn all vertices of G black; this leads to a new graph invariant, the *iteration index of G* – it is the minimum of iteration indices of all minimum zero forcing sets of G . We present some basic properties of the iteration index and discuss some preliminary results on certain graphs.

Key Words: zero forcing set, zero forcing number, iteration index of a zero forcing set, Cartesian product of graphs, bouquet of circles

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1 Introduction

The notion of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced in [1] to bound the minimum rank for numerous families of graphs. Zero forcing parameters were further studied and applied to the minimum rank problem in [2, 3, 5, 6, 7]. In this paper, we introduce and study the iteration index of a zero forcing set in a graph; as we’ll see, this is a very natural graph parameter associated with a minimum zero forcing set of a graph. After the requisite definitions and notations on the graphs to be considered, we’ll give a brief review of the notions and results associated with the zero forcing parameter.

In this paper, a graph $G = (V(G), E(G))$ has no isolated vertices and is finite, simple, and undirected. The *degree* of a vertex v in G is denoted by $\deg_G(v)$, and an *end-vertex* is the vertex of degree one. The minimum degree over all vertices of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph of G induced by S , and we denote by $G - S$ the subgraph of G induced by $V(G) \setminus S$. For any vertex $v \in V(G)$, the *open neighborhood* of v in G , denoted by $N_G(v)$, is the set of all vertices adjacent to v in G , and the *closed neighborhood* of v , denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$; we drop G when ambiguity is not a concern. We denote by P_n , C_n , and K_n the path, the cycle, and the complete graph, respectively, on n vertices. A complete bipartite graph with partite sets having p and q vertices is denoted by $K_{p,q}$. The *path cover number* $P(G)$ of G is the smallest positive integer m such that there are m vertex-disjoint paths P^1, P^2, \dots, P^m in G that cover all vertices of G . The *Cartesian product* of two graphs G and H , denoted by $G \square H$, is the graph with the vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$ or (2) $v = v'$ and $uu' \in E(G)$. For other graph theory terminology, we refer to [4].

Let each vertex of a graph be given either the color black or the color white. Denote by Z the initial set of black vertices. The “color-change rule” changes the color of a vertex w from white to black if the white vertex is the only white neighbor of a black vertex u ; in this case, we may say that u *forces* w and write $u \rightarrow w$. Of course, there may be more than one black vertex capable of forcing w , but we associate only one forcing vertex to w at a time. Applying the color-change rule to all vertices of Z , we obtain an *updated* set of black vertices $Z_1 \supseteq Z$. Clearly, not all vertices in Z need to be forcing vertices, and if a vertex u in Z forces w , then u becomes inactive – i.e., unable to force thereafter. The vertex w replaces u as a potential forcing vertex in Z_1 ; thus, Z_1 has at most $|Z|$ many potentially forcing vertices. Applying the color change rule to Z_1 results in another updated set $Z_2 \supseteq Z_1$ of black vertices. Continuing this process until no more color change is possible, we obtain a nested sequence of sets $Z = Z_0 \subseteq Z_1 \subseteq \dots \subseteq Z_n$. The initial set Z is said to be a *zero forcing set* if $Z_n = V(G)$. A “chronological list of forces” is a record of the forcing actions in the order in which they are performed. Given any chronological list of forces, a “forcing chain” is a sequence u_1, u_2, \dots, u_t such that $u_i \rightarrow u_{i+1}$ for $i = 1, 2, \dots, t - 1$. In consideration of all lists of forces leading from vertices in $Z = Z_0$ to all vertices in $Z_i - Z_{i-1}$, we see that $|Z_i - Z_{i-1}| \leq |Z|$ for each $i \in \{1, 2, \dots, n\}$. The zero forcing number of G , denoted by $Z(G)$, is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

A “maximal forcing chain” is a forcing chain that is not a subsequence of another forcing chain. If Z is a zero forcing set, then a “reversal” of Z is the set of last vertices of maximal forcing chains of a chronological list of forces. The following are some of the known properties of zero forcing parameters:

- ◇ [3] For any graph G , $\delta(G) \leq Z(G)$.
- ◇ [2] If Z is a zero forcing set of a graph G , then any reversal of Z is also a zero forcing set of G .
- ◇ [2] If a graph G has a unique zero forcing set, then G has no edges; i.e., G consists of isolated vertices.
- ◇ [2] For any graph G , $P(G) \leq Z(G)$.
- ◇ [1] For any tree T , $P(T) = Z(T)$.

For two graphs G and H such that $H \subseteq G$, one cannot exactly determine $Z(G)$ from $Z(H)$ – or vice versa, but the following holds.

Theorem 1.1. [5] *Let G be any graph. Then*

- (i) *For $v \in V(G)$, $Z(G) - 1 \leq Z(G - \{v\}) \leq Z(G) + 1$.*
- (ii) *For $e \in E(G)$, $Z(G) - 1 \leq Z(G - e) \leq Z(G) + 1$.*

2 Iteration index of a graph

To facilitate the precise definition of the iteration index $I(G)$ of a graph G , we shall first more precisely (and concisely) define zero forcing parameters in terms of a discrete dynamical system associated with G , which we'll call the zero forcing system.

Definition 2.1. Given any graph G , the zero forcing system induced by a vertex set $S \subseteq V(G)$ is the following recursively defined sequence of functions $\chi_S^i : V(G) \rightarrow \{0, 1\}$ such that

$$\chi_S^0(v) = \begin{cases} 0 & \text{if } v \in S \\ 1 & \text{if } v \in V(G) \setminus S; \end{cases}$$

let χ_S^i be defined for $i \geq 0$, then

$$\chi_S^{i+1}(v) = \begin{cases} 0 & \text{if } \chi_S^i(v) = 0 \\ 0 & \text{if } \chi_S^i(v) = 1, \exists u \in N(v) \text{ such that } \forall w \in N[u] \text{ with } w \neq v, \chi_S^i(w) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.2. A vertex set $Z \subseteq V(G)$ is a *zero forcing set* if there exists some $n \geq 0$ such that $\chi_Z^n(v) = 0, \forall v \in V(G)$. For $i \geq 1$, we define the i -th *derived set* of Z , denoted by D_Z^i , as $D_Z^i = \{v \in V(G) : \chi_Z^i(v) = 0 \text{ and } \chi_Z^{i-1}(v) = 1\}$.

Definition 2.3. The *zero forcing number* $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$. And a zero forcing set of cardinality $Z(G)$ will be called a $Z(G)$ -set.

Next, we define the iteration index of a graph, on which the present paper is focused.

Definition 2.4. For any zero forcing set Z of G , the *iteration index* $I_Z(G)$ of Z is the minimum $n \geq 0$ such that $\chi_Z^n(v) = 0$ for any $v \in V(G)$. And the iteration index of G is $I(G) = \min\{|I_Z(G)| : Z \text{ is a } Z(G)\text{-set}\}$.

Note that Z_i for $i \geq 1$ as defined earlier is $(\chi_Z^i)^{-1}(0) = Z \cup (\cup_{j=1}^i D_Z^j)$. And thus $I_Z(G)$ is the length n of the strictly increasing sequence of sets $Z = Z_0 \subset \dots \subset Z_{n-1} \subset Z_n = V(G)$, where $Z_{n-1} \neq V(G)$. In prose, the iteration index of a zero forcing set Z of G is simply the number of global (taking all black vertices at each step) applications of the color-change rule required to effect all vertices of G black, starting with Z . The minimum ($I(G)$) among such values for all $Z(G)$ -sets is then an invariant of G which is intrinsically interesting. From the “real world” modeling (or discrete dynamical system) perspective, here is a possible scenario: There are initially $|Z|$ persons carrying a certain condition or trait (anything from a virus to a genetic mutation) in a population of $|V(G)|$ people, where the edges $E(G)$ characterize, say, inter-personal relation of a certain type. If Z is capable of passing the condition to the entire population (i.e., “zero forcing”), then $I_Z(G)$ may represent the number of units of time (anything from days to millennia) necessary for the entire population to acquire the condition or trait.

Now, we consider an algebraic interpretation of $I_Z(G)$; it is related to Proposition 2.3 of [1] which states that: “Let Z be a zero forcing set of $G = (V, E)$ and $A \in \mathcal{S}(F, G)$. [Here, $A = (A_{ij})$ is a symmetric matrix where the diagonal entries are arbitrary elements of a field F and, for $i \neq j$, $A_{ij} \neq 0$ exactly when ij is an edge of the graph G .] If $\mathbf{x} \in \ker(A)$ and $\text{supp}(\mathbf{x}) \cap Z = \emptyset$, then $\mathbf{x} = \mathbf{0}$ ” Rephrasing slightly, this says that $x_i = 0$ (where $\mathbf{x} = (x_i)$) for each $i \in Z$ and $A\mathbf{x} = \mathbf{0}$ together imply that $x_j = 0$ for $j \notin Z$ as well. The length of the longest forcing chain of Z ($LLFC(Z)$) appears to be the number of steps needed to reach $\mathbf{x} = \mathbf{0}$ by solving the linear system $A\mathbf{x} = \mathbf{0}$ through *naive substitution*, starting with the data $x_i = 0$ for each $i \in Z$ — as Example 1 will show. It's clear that $I_Z(G)$ is an upper bound for the $LLFC(Z)$. It will be shown in Example 2 that $I_Z(G)$ may be strictly greater than $LLFC(Z)$. However, $I_Z(G)$ has the advantage of being canonically defined,

in contrast to the notion of the forcing chain: After fixing a zero forcing set Z , an arbitrary choice must be made when there are two or more forcing vertices at any given step. Thus, there may be multiple reversals of Z ; a reversal of a reversal of Z is not necessarily Z — to name two of the side effects of the non-canonical nature of the forcing chain.

Example 1. Let $G = C_3 \square K_2$, with vertices labeled as in Figure 1. Notice $Z(G) \geq \delta(G) = 3$. Since

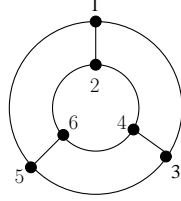


Figure 1: $C_3 \square K_2$

$\{2, 4, 6\}$ is a zero forcing set, we have $Z(G) = 3$. Since $Z(G) < |V(G)|$, we have $I(G) \geq 1$. With $Z = \{2, 4, 6\}$, we get $D_Z^1 = \{1, 3, 5\}$, and so $I_Z(G) = I(G) = 1$.

Now, $Z' = \{3, 4, 6\}$ is also a $Z(G)$ -set. Note that $D_{Z'}^1 = \{2\}$ and $D_{Z'}^2 = \{1, 5\}$. So $I_{Z'}(G) = 2$. Thus, we see that $I_Z(G)$ is not constant as Z varies over $Z(G)$ -sets.

Let A be the generic symmetric matrix associated with the graph in Figure 1; let \mathbf{v} and \mathbf{w} be supported outside of Z and Z' , respectively. Thus

$$A = \begin{pmatrix} a_1 & c_1 & c_2 & 0 & c_3 & 0 \\ c_1 & a_2 & 0 & c_4 & 0 & c_5 \\ c_2 & 0 & a_3 & c_6 & c_7 & 0 \\ 0 & c_4 & c_6 & a_4 & 0 & c_8 \\ c_3 & 0 & c_7 & 0 & a_5 & c_9 \\ 0 & c_5 & 0 & c_8 & c_9 & a_6 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ 0 \\ v_3 \\ 0 \\ v_5 \\ 0 \end{pmatrix}, \text{ and } \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ w_5 \\ 0 \end{pmatrix}$$

where a_i 's are arbitrary and c_j 's are each non-zero real numbers. First suppose $A\mathbf{v} = \mathbf{0}$. Then we have the following system of linear equations:

$$\begin{cases} a_1 \cdot v_1 + c_2 \cdot v_3 + c_3 \cdot v_5 = 0 & (1) \\ c_1 \cdot v_1 = 0 & (2) \\ c_2 \cdot v_1 + a_3 \cdot v_3 + c_7 \cdot v_5 = 0 & (3) \\ c_6 \cdot v_3 = 0 & (4) \\ c_3 \cdot v_1 + c_7 \cdot v_3 + a_5 \cdot v_5 = 0 & (5) \\ c_9 \cdot v_5 = 0. & (6) \end{cases}$$

From the second, fourth, and sixth equations above, we get $v_1 = v_3 = v_5 = 0$. Here, we reach $\mathbf{v} = \mathbf{0}$ in one step, corresponding to $LLFC(Z) = I_Z(G) = 1$.

Next suppose $A\mathbf{w} = \mathbf{0}$. Then we have the following system of linear equations:

$$\begin{cases} a_1 \cdot w_1 + c_1 \cdot w_2 + c_3 \cdot w_5 = 0 & (1) \\ c_1 \cdot w_1 + a_2 \cdot w_2 = 0 & (2) \\ c_2 \cdot w_1 + c_7 \cdot w_5 = 0 & (3) \\ c_4 \cdot w_2 = 0 & (4) \\ c_3 \cdot w_1 + a_5 \cdot w_5 = 0 & (5) \\ c_5 \cdot w_2 + c_9 \cdot w_5 = 0. & (6) \end{cases}$$

From the fourth equation above, we get $w_2 = 0$, which corresponds to $D_{Z'}^1$. By applying $w_2 = 0$ to the system of linear equations above, we get

$$\begin{cases} a_1 \cdot w_1 + c_3 \cdot w_5 = 0 & (1) \\ c_1 \cdot w_1 = 0 & (2) \\ c_2 \cdot w_1 + c_7 \cdot w_5 = 0 & (3) \\ c_3 \cdot w_1 + a_5 \cdot w_5 = 0 & (4) \\ c_9 \cdot w_5 = 0. & (5) \end{cases}$$

The second and fifth equations yield $w_1 = 0 = w_5$, which corresponds to $D_{Z'}^2$. Here, we reach $\mathbf{w} = \mathbf{0}$ in two steps, corresponding to $LLFC(Z') = I_{Z'}(G) = 2$.

Example 2. As discussed in the introduction, the notion of “a forcing chain” has been introduced and made use of in [1] and elsewhere. However, the length of the longest forcing chain of a graph can be strictly less than its iteration index. For example, let T be the tree in Figure 2. Since $P(T) = 3$, $Z(T) = 3$. One can readily check that there are ten $Z(G)$ -sets for T : $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{2, 4, 5\}$, $\{2, 4, 9\}$, $\{2, 5, 9\}$, $\{3, 4, 5\}$, $\{3, 4, 9\}$, and $\{3, 5, 9\}$. Further, one can check for Z (any of the ten sets) that the length of the longest forcing chain is two, while $I_Z(T) = I(T) = 3$. Let A denote the generic symmetric matrix associated with T , and let $\mathbf{v} \in \ker(A)$ be a vector supported outside of Z . Solving $A\mathbf{v} = \mathbf{0}$ through naive substitution starting with the data $v_i = 0$ for each $i \in Z$ as we did in the previous example, one sees that the number of steps needed to reach $\mathbf{v} = \mathbf{0}$ is $LLFC(Z)$ rather than $I(Z)$.

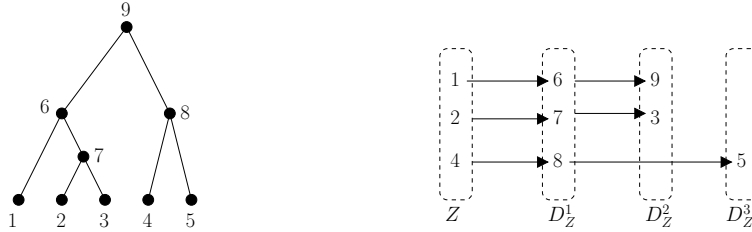


Figure 2: An exmple showing that $LLFC(Z) = 2 < I_Z(G) = I(G) = 3$

Theorem 2.5. *For any graph G that is not edgeless,*

$$\max \left\{ \frac{|V(G)|}{Z(G)} - 1, 1 \right\} \leq I(G) \leq |V(G)| - Z(G).$$

Proof. Since G has an edge, the cardinality of a minimum zero forcing set Z_0 is less than $|V(G)|$; thus, $I(G) \geq 1$. Application of color-change rule results in a chain of sets $Z_0 \subset Z_1 \subset \dots \subset Z_k = V(G)$, where $k = I(G)$. Note that $|Z_0| = Z(G)$. Since $V(G)$ is the disjoint union of Z_0 and $Z_i - Z_{i-1}$ for $1 \leq i \leq k$, we have

$$|V(G)| = |Z_0| + \sum_{i=1}^k |Z_i - Z_{i-1}| \leq |Z_0| + k|Z_0| = (k+1)|Z_0|.$$

So $\frac{|V(G)|}{|Z_0|} - 1 \leq k = I(G)$. The inequality $I(G) \leq |V(G)| - |Z_0|$ is trivial since $Z_i - Z_{i-1} \neq \emptyset$ for $1 \leq i \leq k$. This completes the proof. \square

Remark 1. The bounds in Theorem 2.5 are best possible. We illustrate the sharpness of the lower bound with two examples. First notice $I(K_n) = 1$ (see (i) of Observation 2.6). Second, let G be $P_s \square P_t$: for $s \geq t \geq 2$, one can readily check that $|V(G)| = st$, $Z(G) = t$, and $I(G) = s - 1 = \frac{|V(G)|}{Z(G)} - 1$. For the sharpness of the upper bound, K_n again serves as an example. As a less trivial example, let G be the graph obtained by joining the center of a star $K_{1,m}$ to an

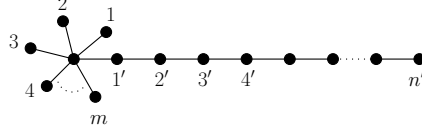


Figure 3: A graph G satisfying $I(G) = |V(G)| - Z(G)$

end-vertex of the path P_n (see Figure 3): one can readily check that $|V(G)| = m + n + 1$, $Z(G) = m$, and $I(G) = n + 1 = |V(G)| - Z(G)$.

In [1], the zero forcing number for a cycle, a path, a complete graph, and a complete bipartite graph (respectively) was obtained; i.e., (i) $Z(P_n) = 1$ for $n \geq 2$; (ii) $Z(C_n) = 2$ for $n \geq 3$, (iii) $Z(K_n) = n - 1$ for $n \geq 2$, and (iv) $Z(K_{p,q}) = p + q - 2$, where $p, q \geq 2$.

Observation 2.6.

- (i) $I(K_n) = 1$ for $n \geq 2$, since $Z(K_n) = n - 1$.
- (ii) $I(P_n) = n - 1$ for $n \geq 2$, since $Z(P_n) = 1$ and only an end-vertex is a minimum zero forcing set.
- (iii) $I(C_n) = \lceil \frac{n-2}{2} \rceil$ for $n \geq 3$, since $Z(C_n) = 2$ and only an adjacent pair of vertices is a minimum zero forcing set.
- (iv) $I(K_{1,q}) = 2$ for $q \geq 2$, since $Z(K_{1,q}) = q - 1$ and any minimum zero forcing set must omit the central vertex along with an end-vertex.
- (v) $I(K_{p,q}) = 1$ for $p, q \geq 2$, since $Z(K_{p,q}) = p + q - 2$ and one may choose the minimum zero forcing set that omits a vertex from each partite set.

3 Zero forcing number and Iteration index of the Cartesian product of some graphs

Consider $G \square H$ with $|G| = s$ and $|H| = t$. Let the t copies of G to be $G^{(1)}, G^{(2)}, \dots, G^{(t)}$ from the left to the right and let the s copies of H to be $H^{(1)}, H^{(2)}, \dots, H^{(s)}$ from the top to the bottom. The vertex labeled (x, y) in $G \square H$ is the result of the intersection of $G^{(y)}$ and $H^{(x)}$. See Figure 4 for $G = P_s$ and $H = P_t$. The Cartesian product $P_s \square P_t$ is also called a *grid graph*.

In [1], it is shown that $Z(P_s \square P_t) = \min\{s, t\}$ for $s, t \geq 2$, $Z(C_s \square P_t) = \min\{s, 2t\}$ for $s \geq 3, t \geq 2$, $Z(K_s \square P_t) = s$ for $s, t \geq 2$, $Z(K_s \square K_t) = st - s - t + 2$ for $s, t \geq 2$, and $Z(C_s \square K_t) = 2t$ for $s \geq 4$.

Theorem 3.1.

- (i) For $t \geq s \geq 2$, $I(P_s \square P_t) = t - 1$.
- (ii) For $s, t \geq 2$, $I(K_s \square P_t) = t - 1$.
- (iii) For $s \geq 3$ and $t \geq 2$, $I(C_s \square P_t) = \begin{cases} \lceil \frac{s-2}{2} \rceil & \text{if } s \geq 2t \\ t - 1 & \text{if } s < 2t. \end{cases}$
- (iv) For $s \geq 4$ and $t \geq 2$, $I(C_s \square K_t) = \lceil \frac{s-2}{2} \rceil$.

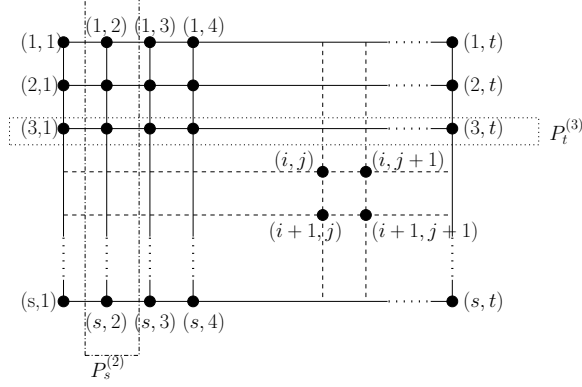


Figure 4: The grid graph $P_s \square P_t$

Proof. (i) Since $Z(P_s \square P_t) = s$, $I(P_s \square P_t) \geq t-1$ by Theorem 2.5. If we take $Z_0 = \{(1, 1), (2, 1), \dots, (s, 1)\}$ as a $Z(P_s \square P_t)$ -set (see Figure 4), then, for each $1 \leq i \leq s$, $(i, j) \rightarrow (i, j+1)$ if $1 \leq j \leq t-1$. Hence $Z_{t-1} = V(P_s \square P_t)$, so $I(P_s \square P_t) \leq t-1$.

(ii) Since $Z(K_s \square P_t) = s$, $I(K_s \square P_t) \geq t-1$ by Theorem 2.5. If we take $Z_0 = \{(1, 1), (2, 1), \dots, (s, 1)\}$ as a $Z(K_s \square P_t)$ -set (see Figure 5 for $K_5 \square P_3$), then, for each $1 \leq i \leq s$, $(i, j) \rightarrow (i, j+1)$ if $1 \leq j \leq t-1$. Hence $Z_{t-1} = V(K_s \square P_t)$, so $I(K_s \square P_t) \leq t-1$.

(iii) We consider two cases.

Case 1. $s \geq 2t$: Then $Z(C_s \square P_t) = 2t$, and $I(C_s \square P_t) \geq \lceil \frac{s-2}{2} \rceil$ by Theorem 2.5. By taking $Z_0 = \{(1, 1), (1, 2), \dots, (1, t)\} \cup \{(s, 1), (s, 2), \dots, (s, t)\}$ as a $Z(C_s \square P_t)$ -set, we have, for each $1 \leq j \leq t$, that $(i, j) \rightarrow (i+1, j)$ for $1 \leq i \leq \lceil \frac{s}{2} \rceil - 1$ and $(i, j) \rightarrow (i-1, j)$ for $\lfloor \frac{s}{2} \rfloor + 2 \leq i \leq s$. Hence $Z_{\lceil \frac{s-2}{2} \rceil} = V(C_s \square P_t)$, so $I(C_s \square P_t) \leq \lceil \frac{s-2}{2} \rceil$. Thus $I(C_s \square P_t) = \lceil \frac{s-2}{2} \rceil$.

Case 2. $s < 2t$: Then $Z(C_s \square P_t) = s$, and $I(C_s \square P_t) \geq t-1$ by Theorem 2.5. If we take $Z_0 = \{(1, 1), (2, 1), \dots, (s, 1)\}$ as a $Z(C_s \square P_t)$ -set, then, for each $1 \leq i \leq s$, $(i, j) \rightarrow (i, j+1)$ if $1 \leq j \leq t-1$. Hence $Z_{t-1} = V(C_s \square P_t)$, so $I(C_s \square P_t) \leq t-1$. Thus $I(C_s \square P_t) = t-1$.

(iv) Since $Z(C_s \square K_t) = 2t$, $I(C_s \square K_t) \geq \lceil \frac{s-2}{2} \rceil$ by Theorem 2.5. By taking $Z_0 = \{(1, 1), (1, 2), \dots, (1, t)\} \cup \{(s, 1), (s, 2), \dots, (s, t)\}$ as a $Z(C_s \square K_t)$ -set, we have, for each $1 \leq j \leq t$, that $(i, j) \rightarrow (i+1, j)$ for $1 \leq i \leq \lceil \frac{s}{2} \rceil - 1$ and $(i, j) \rightarrow (i-1, j)$ for $\lfloor \frac{s}{2} \rfloor + 2 \leq i \leq s$. Hence $Z_{\lceil \frac{s-2}{2} \rceil} = V(C_s \square K_t)$, so $I(C_s \square K_t) \leq \lceil \frac{s-2}{2} \rceil$. \square

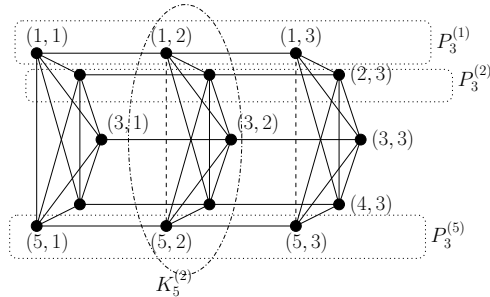


Figure 5: The Cartesian product $K_5 \square P_3$

Next we consider $K_s \square K_t$ (see Figure 6 for $K_5 \square K_4$). By (i) and (ii) of Theorem 3.1, we have $I(K_2 \square K_2) = I(P_2 \square P_2) = 1$ and $I(K_3 \square K_2) = I(K_3 \square P_2) = 1$.

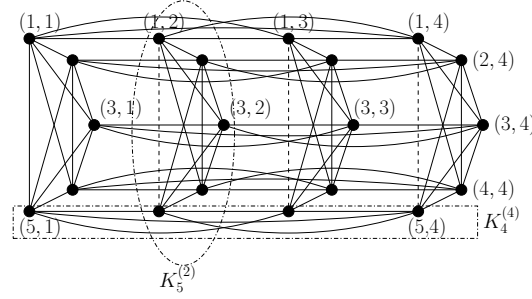


Figure 6: The Cartesian product $K_5 \square K_4$

Theorem 3.2. For $s, t \geq 3$, $I(K_s \square K_t) = 2$.

Proof. Since $Z(K_s \square K_t) = st - s - t + 2 < |V(K_s \square K_t)|$, $I(K_s \square K_t) \geq 1$. Assume, for the sake of contradiction, that $I(K_s \square K_t) = 1$. Take any $Z(K_s \square K_t)$ -set Z with $I_Z(K_s \square K_t) = 1$ and let w_1, w_2, \dots, w_x be an ordered listing of all the vertices not in Z ; i.e, the “white vertices”. Without loss of generality (WLOG), let $t \geq s \geq 3$. Let w_1 be located in the i -th row **and** j -th column (see Figure 6). Since $I_Z(K_s \square K_t) = 1$, there must exist a “black” vertex $b \in Z$ located in the i -th row **or** j -th column such that each $v \in N[b] \setminus \{w_1\}$ is in Z , and b forces w_1 . Thus, w_1 implies the existence of $s+t-2$ black vertices. Likewise, each w_i implies the existence of (not counting overlaps) $s+t-2$ black vertices – namely a “black row” and a “black column”, disregarding w_i itself. Having considered all the black rows and black columns (disregarding the w_i ’s) corresponding to vertices w_1 through w_q for $1 \leq q < x$, consider w_{q+1} . Notice that either the black row or the black column (again, disregarding w_{q+1} itself) corresponding to w_{q+1} must be “new”, since either the corresponding row or corresponding column contains w_{q+1} : this means that w_{q+1} implies the existence of at least $s-1$ new black vertices. We thus have the inequality $s+t-2 + (x-1)(s-1) \leq st - s - t + 2$, which easily implies that $x < t$, contradicting the fact $x = t + s - 2$ and the hypothesis $t \geq s \geq 3$. Hence $I(K_s \square K_t) \geq 2$.

On the other hand, if we take $Z_0 = (\cup_{j=1}^{t-1} \{(2, j), (3, j), \dots, (s, j)\}) \cup \{(1, 1)\}$ as a $Z(K_s \square K_t)$ -set with $|Z_0| = st - s - t + 2$, then, for each $2 \leq i \leq s$, $(i, 1) \rightarrow (i, t)$; so $Z_1 = Z_0 \cup \{(2, t), (3, t), \dots, (s, t)\} = V(K_s \square K_t) \setminus \{(1, 2), (1, 3), \dots, (1, t)\}$. Next, for each $2 \leq j \leq t$, $(s, j) \rightarrow (1, j)$, and thus $Z_2 = V(K_s \square K_t)$. Therefore, $I(K_s \square K_t) \leq 2$. \square

Remark 2. Noticing $Z(C_3 \square K_t) = Z(K_3 \square K_t) = 2t - 1$, we have $I(C_3 \square K_2) = I(C_3 \square P_2) = 1$ by (iii) of Theorem 3.1, and $I(C_3 \square K_t) = 2$ for $t \geq 3$ by Theorem 3.2.

We recall that

Proposition 3.3. ([1], Prop. 2.5) For any graphs G and H , $Z(G \square H) \leq \min\{Z(G)|H|, Z(H)|G|\}$.

Proposition 3.4. Let $t > s \geq 3$. Then

$$\begin{cases} Z(C_s \square C_s) \leq 2s - 1 & \text{if } s \text{ is odd} & (1) \\ Z(C_s \square C_s) \leq 2s & \text{if } s \text{ is even} & (2) \\ Z(C_s \square C_t) \leq 2s & & (3). \end{cases}$$

Proof. Since $Z(C_m) = 2$ for any $m \geq 3$, parts (2) and (3) of the conclusion follow immediately from Proposition 3.3; so we only need to show part (1) of the conclusion. It’s obvious that $2s$ many black

vertices on two adjacent cycles (C_s) form a zero forcing set. It thus suffices to show that starting with $2s - 1$ black vertices on two adjacent cycles, after finitely many applications of the color-change rule, one obtains two adjacent cycles as a subset of the set of black vertices. This can be seen as follows: Label the s^2 vertices on $C_s \square C_s$ by (i, j) , where $1 \leq i, j \leq s$. Take as the initial set of black vertices $\{(i, j) : 1 \leq i \leq 2 \text{ and } 1 \leq j \leq s\} \setminus \{(1, \frac{s+1}{2})\}$ (recall that s is odd). One can readily check (see Figure 7) that after $\frac{s-1}{2}$ applications of the color change rule, the two adjacent cycles $\{(i, j) : 1 \leq i \leq s \text{ and } j \in \{1, s\}\}$ will consist of only black vertices. \square

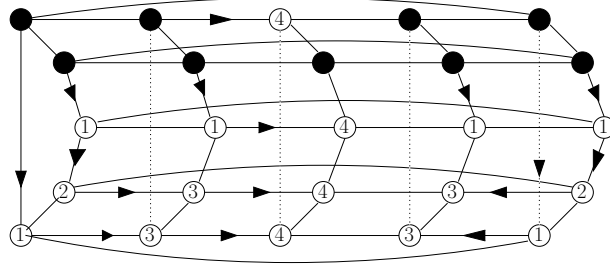


Figure 7: The set of black vertices is a zero forcing set Z_0 of $C_5 \square C_5$, the number m in each vertex indicates that the vertex is in Z_m , and the arrows indicate possible forcing chains corresponding to Z_0 .

Remark 3. Note that $Z(C_3 \square C_3) = Z(K_3 \square K_3) = 5$ and $I(C_3 \square C_3) = I(K_3 \square K_3) = 2$ by Theorem 3.2. Also note that $Z(C_3 \square C_4) = Z(K_3 \square C_4) = 6$ and $I(C_3 \square C_4) = I(K_3 \square C_4) = \lceil \frac{4-2}{2} \rceil = 1$ by (iv) of Theorem 3.1. Further, one can check that $Z(C_4 \square C_4) = 8$ and $I(C_4 \square C_4) = 1$. Thus we have the following

Conjecture. Let $t > s \geq 3$. Then

$$\begin{cases} Z(C_s \square C_s) = 2s - 1 & \text{if } s \text{ is odd} \\ Z(C_s \square C_s) = 2s & \text{if } s \text{ is even} \\ Z(C_s \square C_t) = 2s. \end{cases}$$

4 Upper bounds of iteration index of triangular grids and king grids

The *triangular grid graph*, denoted by $P_s \boxtimes P_t$, can be obtained from the grid graph $P_s \square P_t$ by adding a diagonal edge of negative slope to each C_4 square. In [5], it was shown that $Z(P_s \boxtimes P_t) = s$ if $t \geq s \geq 2$. Figure 8 shows a zero forcing set of $P_4 \boxtimes P_{10}$ and its forcing chain, where the set of black vertices is a zero forcing set Z_0 and the number m in each vertex indicates that the vertex is in Z_m .

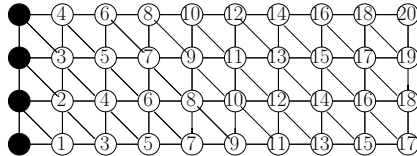


Figure 8: The 4×10 triangular grid graph and its zero forcing chain

Theorem 4.1. For $t \geq s \geq 2$, $I(P_s \boxtimes P_t) \leq 2t + s - 4$.

Proof. Refer to (A) of Figure 9 for the labeling of vertices. It's known that $Z(P_s \boxtimes P_t) = s$ for $2 \leq s \leq t$. Take $Z_0 = \{(0,0), (0,1), \dots, (0, s-1)\}$. We'll show that the vertex $(i,j) \in Z_{2i+j-1}$ for $(i,j) \notin Z_0$: the theorem would then follow since the range of the function $n = n(i,j) = 2i + j - 1$ over the lattice $\Lambda = \{1, 2, \dots, t-1\} \times \{0, 1, \dots, s-1\}$ for $2 \leq s \leq t$ is the set $\{1, 2, \dots, 2t + s - 4\}$. We'll induct on $n \in \{1, 2, \dots, 2t + s - 4\}$.

We prove by strong induction. Let $n = 1$. The only solution to $2i + j - 1 = 1$ for $(i,j) \in \Lambda$ is $(1,0)$. One sees immediately that $(1,0) \in Z_1$, since it's forced by $(0,0) \in Z_0$.

Suppose $(i,j) \in Z_{n=2i+j-1}$ for all $(i,j) \in \Lambda$ such that $1 \leq 2i + j - 1 < n_0$, where $2 \leq n_0 \leq 2t + s - 4$. We need to show, for (i,j) with $2i + j - 1 = n_0$, that $(i,j) \in Z_{n_0}$. Now, $(i,j) \in Z_{n_0}$ ("white vertex") (i,j) is turned "black" in or before the n_0 -th iteration) if (i,j) has a neighbor ("black vertex") (i',j') such that $(x,y) \in N[(i',j')] \setminus \{(i,j)\}$ implies $2x + y - 1 \leq n_0 - 1$ (i.e., the vertex (x,y) has been turned "black" in or before the $(n_0 - 1)$ -th iteration). We claim that the vertex (i',j') may be taken to be $(i-1,j)$: (B) of Figure 9 shows the local picture where $|N[(i-1,j)]| = 7$, the maximum possible; it's trivially checked that $2x + y - 1 < n_0$ for any $(x,y) \in N[(i-1,j)] \setminus \{(i,j)\}$. \square

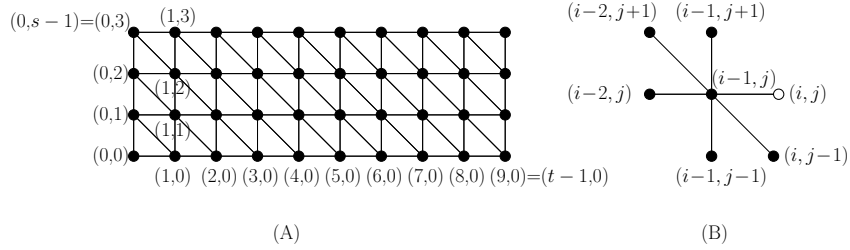


Figure 9: The labeling of 4×10 triangular grid graph and $N[(i-1,j)]$

The *king grid graph*, denoted by $P_s \boxtimes P_t$, can be obtained from the grid graph $P_s \square P_t$ by adding both diagonal edges to each C_4 square. In [1], it was shown that $Z(P_s \boxtimes P_t) = s + t - 1$ for $s, t \geq 2$. Figure 10 shows $P_4 \boxtimes P_{10}$ and $P_3 \boxtimes P_{10}$, along with a zero forcing set and its forcing chain for each graph.

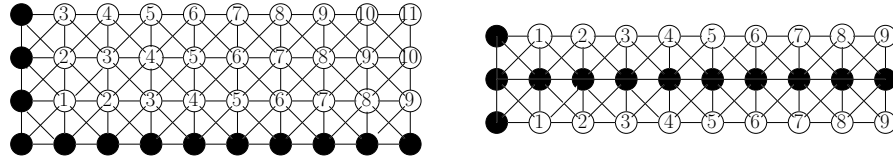


Figure 10: $P_4 \boxtimes P_{10}$ and $P_3 \boxtimes P_{10}$, together with a zero forcing set for each graph: the number m in each vertex indicates that the vertex is in Z_m .

Theorem 4.2. For $s, t \geq 2$, $I(P_s \boxtimes P_t) \leq s + t - 3$. In fact, $I(P_3 \boxtimes P_t) \leq t - 1$ for $t \geq 2$.

Proof. Refer to Figure 11 for the labeling of vertices. Noting $Z(P_s \boxtimes P_t) = s + t - 1$, let $Z_0 = (\cup_{j=0}^{s-1} \{(0,j)\}) \cup (\cup_{i=1}^{t-1} \{(i,0)\})$. We'll show that the vertex $(i,j) \in Z_{n=i+j-1}$ for $(i,j) \notin Z_0$: the first assertion of the theorem would then follow since the range of the function $n = n(i,j) = i + j - 1$ over the lattice $\Lambda = \{1, 2, \dots, t-1\} \times \{1, \dots, s-1\}$ for $s, t \geq 2$ is the set $\{1, 2, \dots, s + t - 3\}$. We'll induct on $n \in \{1, 2, \dots, s + t - 3\}$.

We prove by strong induction. Let $n = 1$. The only solution to $i + j - 1 = 1$ for $(i,j) \in \Lambda$ is $(1,1)$. One sees immediately that $(1,1) \in Z_1$, since it's forced by $(0,0) \in Z_0$.

Suppose $(i, j) \in Z_{n_0-1}$ for all $(i, j) \in \Lambda$ such that $1 \leq i + j - 1 \leq n_0 - 1$, where $2 \leq n_0 \leq s + t - 3$. We need to show, for (i, j) with $i + j - 1 = n_0$, that $(i, j) \in Z_{n_0}$. Notice $(i - 1, j - 1)$ is adjacent to (i, j) , and it suffices to show that $(x, y) \in N[(i - 1, j - 1)] \setminus \{(i, j)\}$ implies $x + y - 1 < n_0$. But this is obvious — in view of the coordinates assigned to the vertices.

Next, consider the particular case of $P_3 \boxtimes P_t$ for $t \geq 2$. If we take $Z_0 = (\cup_{i=0}^{t-1} \{(i, 1)\}) \cup \{(0, 0), (0, 2)\}$ with $|Z_0| = t + 2$, then, for each $0 \leq i \leq t - 2$, we have that $(i, 0) \rightarrow (i + 1, 0)$ and $(i, 2) \rightarrow (i + 1, 2)$. Hence $Z_{t-1} = V(P_3 \boxtimes P_t)$; i.e., $I(P_3 \boxtimes P_t) \leq t - 1$. \square

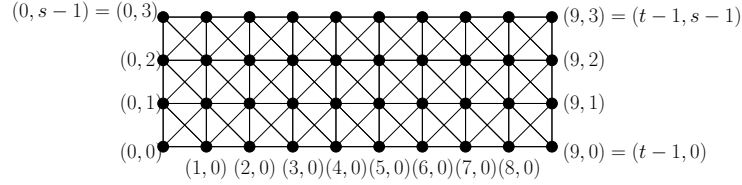


Figure 11: The king grid graph $P_4 \boxtimes P_{10}$

5 Zero forcing number and iteration index of a bouquet of circles

The bouquet of circles — the figure 8, in particular — has been studied as a motivating example to introduce the fundamental group on a graph (see p.189, [9]). More recently, Llibre and Todd [8], for instance, studied a class of maps on a bouquet of circles from a dynamical system perspective.

For $2 \leq k_1 \leq k_2 \leq \dots \leq k_n$, let $B_n = (k_1, k_2, \dots, k_n)$ be a bouquet of $n \geq 2$ circles C^1, C^2, \dots, C^n , with the cut-vertex v , where k_i is the number of vertices of $C^i - \{v\}$ ($1 \leq i \leq n$). (The $n = 1$ case has already been addressed.) Let $V(C^i) = \{v, w_{i,1}, w_{i,2}, \dots, w_{i,k_i}\}$ such that $vw_{i,1} \in E(B_n)$ and $vw_{i,k_i} \in E(B_n)$, and let the vertices in C^i be cyclically labeled, where $1 \leq i \leq n$. See Figure 12 for $B_3 = (2, 3, 4)$. Note that $|V(B_n)| = 1 + \sum_{i=1}^n k_i$, where $k_i \geq 2$.

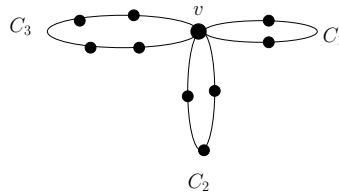


Figure 12: A bouquet of three circles, $B_3 = (2, 3, 4)$

Theorem 5.1. *Let $B_n = (k_1, k_2, \dots, k_n)$ be a bouquet of n circles with cut-vertex v . Then $Z(B_n) = n + 1$.*

Proof. One can readily check that $\{v\} \cup \{w_{i,1} \mid 1 \leq i \leq n\}$ form a zero forcing set for B_n , and thus $Z(B_n) \leq n + 1$. To prove the theorem, we need to show $Z(B_n) \geq n + 1$. We make the following claims.

Claim 1. At least one vertex from each $C^i - \{v\}$ ($1 \leq i \leq n$) belongs to a $Z(B_n)$ -set.

Proof of Claim 1: This is clearly true by the assumption that $k_i \geq 2$ for each i – one black vertex (namely v) on C^i can not “force”.

Claim 2. Any $Z(B_n)$ -set contains a pair of adjacent vertices on a C^i for some i .

Proof of Claim 2: This is because the degree of every vertex is at least two, and a set of isolated black vertices can not force.

By Claims 1 and 2, we have $Z(B_n) \geq n + 1$. □

Theorem 5.2. Let $B_n = (k_1, k_2, \dots, k_n)$ be a bouquet of n circles with the cut-vertex v , where $n \geq 2$ and $k_i \geq 2$ ($1 \leq i \leq n$). Then $I(B_n) = \left\lceil \frac{k_n + k_{n-1}}{2} \right\rceil - 1$.

Proof. Let Z_0 be a $Z(B_n)$ -set. First, assume that Z_0 contains the cut vertex v . We show $I_{Z_0}(B_n) = \left\lceil \frac{k_n + k_{n-1}}{2} \right\rceil - 1$. Of course, $I(B_n) \leq I_{Z_0}(B_n)$ by definition.

Notice that there is a unique $Z(B_n)$ -set containing the cut-vertex v up to isomorphism of graphs. We take $Z_0 = \{v\} \cup \{w_{i,1} \mid 1 \leq i \leq n\}$. The presence of v in Z_0 ensures that the entire bouquet will be turned black as soon as the vertices in the longest cycle C^n are turned black. If $k_n \leq k_{n-1} + 1$, then the white vertices of C^n are turned black one at a time, and thus $I_{Z_0}(B_n) = k_n - 1 = \left\lceil \frac{k_n + k_{n-1}}{2} \right\rceil - 1$. If $k_n \geq k_{n-1} + 2$, then the white vertices of C^n are turned black one at a time until $(k_{n-1} - 1)$ -th step and two at a time thereafter. This means that $V(B_n) \setminus \{w_{n,(k_{n-1})+1}, w_{n,(k_{n-1})+2}, \dots, w_{n,k_n}\}$ belongs to $Z_{(k_{n-1})-1}$, and $1 \leq |Z_{x+1} - Z_x| \leq 2$ for $x \geq k_{n-1}$. ($|Z_{x+1} - Z_x|$ may be less than 2 only if $x + 1 = I_{Z_0}(B_n)$.) Thus, $I_{Z_0}(B_n) = k_{n-1} - 1 + \left\lceil \frac{k_n - k_{n-1}}{2} \right\rceil = \left\lceil \frac{k_n + k_{n-1}}{2} \right\rceil - 1$.

Second, we show that if Z_0 does not contain v , then $I_{Z_0}(B_n) \geq \left\lceil \frac{k_n + k_{n-1}}{2} \right\rceil - 1$.

By Claims 1 and 2 in the proof of Theorem 5.1, we have $2 \leq |Z_0 \cap V(C^{n-1} \cup C^n)| \leq 3$. We note that the entire bouquet will not be turned black until all vertices in $C^{n-1} \cup C^n$ are turned black. If $|Z_0 \cap V(C^{n-1} \cup C^n)| = 2$, then $|Z_0 \cap V(C^{n-1})| = |Z_0 \cap V(C^n)| = 1$, and forcing on $C^{n-1} \cup C^n$ can not start until v is turned black. Since $v \in Z_m$ for some $m \geq 1$, by the same argument as in the upper bound case, we have $I_{Z_0}(B_n) \geq m + \left\lceil \frac{k_n + k_{n-1}}{2} \right\rceil - 1 \geq \left\lceil \frac{k_n + k_{n-1}}{2} \right\rceil$. We can thus assume that $|Z_0 \cap V(C^{n-1} \cup C^n)| = 3$. Observe that the lower bound is proved if we show that at most two vertices in $\langle V(C^{n-1} \cup C^n) \rangle$ are turned black at a time.

WLOG, assume $|Z_0 \cap C^{n-1}| = 2$, as the other case where $|Z_0 \cap C^n| = 2$ is very similar. Then two vertices in C^{n-1} are turned black at each step and no forcing occurs in C^n until the cut-vertex v is turned black. Now, if v is the last vertex on C^{n-1} to be turned black, then it's clear that at most two vertices in $\langle V(C^{n-1} \cup C^n) \rangle$ are turned black at a time – since, obviously, with any $Z(B_n)$ -set at most two vertices are turned black on any cycle C^i at each step. On the other hand, if v is turned black before C^{n-1} is turned entirely black, then v , being a neighbor to at least two white vertices, can not force until C^{n-1} is turned entirely black. We've thus shown that $Z_m \cap V(C^{n-1} \cup C^n)$ contains at most two forcing vertices for any m or, equivalently, at most two vertices in $\langle V(C^{n-1} \cup C^n) \rangle$ are turned black at a time. □

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References

- [1] AIM Minimum Rank - Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, **428/7** (2008) 1628-1648.

- [2] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, Zero forcing parameters and minimum rank problems, *preprint* (2010).
- [3] A. Berman, S. Friedland, L. Hogben, U. G. Rothblum, and B. Shader, An upper bound for the minimum rank of a graph. *Linear Algebra Appl.*, **429** (2008) 1629-1638.
- [4] G. Chartrand and P. Zhang, *Introduction to Graph Theory*. McGraw-Hill, Kalamazoo, MI (2004).
- [5] C. J. Edholm, L. Hogben, M. Hyunh, J. LaGrange, and D. D. Row, Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. *preprint* (2010).
- [6] L. Huang, G. J. Chang, and H. Yeh, On minimum rank and zero forcing sets of a graph. *Linear Algebra and its Applications*, **432** (2010) 2961-2973.
- [7] C. R. Johnson and A. Leal Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear Multilinear Algebra*, **46** (1999) 139-144.
- [8] J. Llibre and M. Todd, Periods, Lefschetz numbers and entropy for a class of maps on a bouquet of circles, arXiv:math/0409361v1 [math.DS]
- [9] W. S. Massey, *Algebraic Topology: An Introduction*. Graduate Texts in Mathematics **56**, Springer-Verlag (1989)